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The bundles of algebraic and Dirac–Hestenes spinor fields

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Our main objective in this paper is to clarify the *ontology* of Dirac–Hestenes spinor fields (DHSF) and its relationship with even multivector fields, on a Riemann–Cartan spacetime (RCST) $\mathfrak{M}=(M,g,\nabla,\tau_g,\uparrow)$ admitting a spin structure, and to give a mathematically rigorous derivation of the so-called Dirac–Hestenes equation (DHE) in the case where $\mathfrak{M}$ is a Lorentzian spacetime (the general case when $\mathfrak{M}$ is a RCST will be discussed in another publication). To this aim we introduce the Clifford bundle of multivector fields $(\mathcal{C}\ell(M,g))$ and the *left* $(\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M))$ and *right* $(\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M))$ spin-Clifford bundles on the spin manifold (M,g). The relation between *left ideal algebraic spinor fields* (LIASF) and Dirac–Hestenes *spinor* fields (both fields are sections of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$) is clarified. We study in detail the theory of covariant derivatives of Clifford fields as well as that of left and right spin-Clifford fields. A consistent Dirac equation for a DHSF $\Psi\in\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ (denoted $\mathrm{DE}\mathcal{C}\ell^{l}$) on a Lorentzian spacetime is found. We also obtain a *representation* of the $\mathrm{DE}\mathcal{C}\ell^{l}$ in the Clifford bundle $\mathcal{C}\ell(M,g)$. It is such equation that we call the DHE and it is satisfied by Clifford fields $\psi_{\Xi}\in\sec\mathcal{C}\ell(M,g)$. This means that to each DHSF $\Psi\in\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ and spin frame $\Xi\in\sec P_{\mathrm{Spin}^e_{1,3}}(M)$, there is a well-defined sum of even multivector fields $\psi_{\Xi}\in\sec\mathcal{C}\ell(M,g)$ (EMFS) associated with Ψ. Such an EMFS is called a *representative* of the DHSF on the given spin frame. And, of course, such a EMFS (the representative of the DHSF) is *not* a spinor field. With this crucial distinction between a DHSF and its *representatives* on the Clifford bundle, we provide a consistent theory for the covariant derivatives of Clifford and spinor fields of all kinds. We emphasize that the $\mathrm{DE}\mathcal{C}\ell^{l}$ and the DHE, although related, are equations of different mathematical natures. We study also the local Lorentz invariance and the electromagnetic gauge invariance and show that only for the DHE such transformations are of the same mathematical nature, thus suggesting a possible link between them. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1757038]

I. INTRODUCTION

The main objective of this paper is to clarify the *ontology* of Dirac–Hestenes spinor fields (DHSF) (for the genesis of these objects we quote Ref. 19) on general Riemann–Cartan spacetimes (RCST) and to give a mathematically justified account of the Dirac–Hestenes equation

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(DHE) on Lorentzian spacetimes, subjects that have been a matter of many misunderstandings and controversies (as discussed in Ref. 34). Recall that the flat spacetime DHE represents the state of an electron by a map Ψ with values in the even part of the Clifford algebra $\mathbb{R}_{1,3}$. However, a covariant formulation of the DHE on a (possibly curved) Lorentzian spacetime cannot promote Ψ, in a canonical way, to a section of the Clifford bundle $\mathcal{C}\ell(M,g)$ (whose objects transform as tensors and therefore cannot describe spin-1/2 particles). In Ref. 34, DHSF on a Minkowski spacetime were defined as equivalence classes of Clifford fields. Here we follow a different approach, and define DHSF as even sections of an appropriate spinorial Clifford bundle. The objects satisfying the Dirac–Hestenes equation are then even multivector fields which are *representatives* of DHSF on the tensorial Clifford bundle. Moreover, such a representative is manifestly spin-frame dependent, so that no contradiction arises in representing spinors by Clifford fields.

To achieve our goals, we introduce in Sec. II the Clifford bundle of multivector fields $(\mathcal{C}\ell(M,g))$, and the *left* $(\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M))$ and *right* $(\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M))$ spin-Clifford bundles on the spin manifold (M,g), and study in detail how these bundles are related. [Of course, all the results of the present paper could also be obtained in the case where $\mathcal{C}\ell(M,g)$ is a Clifford bundle of nonhomogeneous differential forms.] Left algebraic spinor fields and Dirac–Hestenes spinor fields [both fields are sections of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$] are defined and the relation between them is established. In Sec. IV, a consistent Dirac equation for a DHSF $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ (denoted DE$\mathcal{C}\ell^l$) on a Lorentzian manifold is found. In Sec. V, we obtain a *representation* of the DE$\mathcal{C}\ell^l$ in the Clifford bundle, an equation we call the Dirac–Hestenes equation (DHE), which is satisfied by Clifford fields $\psi_\Xi \in \sec \mathcal{C}\ell(M,g)$. This means that to each DHSF $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ and to each *spin frame* $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ there is a well-defined sum of even multivector fields $\psi_\Xi \in \sec \mathcal{C}\ell(M,g)$ (EMFS) associated with Ψ. Such an EMFS is called a *representative* of the DHSF on the given spin frame. And, of course, such an EMFS (the representative of the DHSF) is *not* a spinor field. With this crucial distinction between a DHSF and their EMFS representatives, we present in Sec. V an *effective* spinorial connection for the representatives of a DHSF on $\mathcal{C}\ell(M,g)$, thus providing a consistent theory for the covariant derivatives of Clifford and spinor fields of all kinds.

We emphasize that the DE$\mathcal{C}\ell^l$ and the DHE, although related, are of different mathematical natures. This issue has been particularly scrutinized in Secs. IV and V. We study also the local Lorentz invariance and the electromagnetic gauge invariance and show that only for the DHE such transformations are of the same mathematical nature, thus suggesting a possible link between them. In a sequel paper we are going to investigate this issue and also (a) the formulation of the DE$\mathcal{C}\ell$ and DHE in an arbitrary Riemann–Cartan spacetime through the use of a variational principle (we shall use in our approach to the subject the techniques of the multivector and extensor calculus developed in Refs. 12–14, 25–28); (b) the theory of the Lie derivative of the LIASF and DHSF; and (c) the claim in Ref. 17 that the existence of spinor fields in a Lorentzian manifold requires a minimum amount of curvature. This problem is important in view of the proposed teleparallel theories of the gravitational field.

Finally, in the Appendix we derive some formulas employed in the main text for the covariant derivative of Clifford and spinor fields, using the general theory of covariant derivatives on associated vector bundles. In general, our notation corresponds to that in Ref. 34.

A few acronyms are used in the present paper (to avoid long sentences) and they are summarized below for the reader's convenience:

DHE—Dirac–Hestenes Equation
DHSF—Dirac–Hestenes Spinor Field
DE$\mathcal{C}\ell^l$—Dirac equation for a DHSF $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$
EMFS—Even Multivector Fields
LIASF—Left Ideal Algebraic Spinor Field
PFB—Principal Fiber Bundle

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RIASF—Right Ideal Algebraic Spinor Field
RCST—Riemann–Cartan Spacetime

II. THE CLIFFORD BUNDLE OF SPACETIME AND THEIR IRREDUCIBLE MODULE REPRESENTATIONS

A. The Clifford bundle of spacetime

Let M be a four dimensional, real, connected, paracompact and noncompact manifold. Let TM $[T^*M]$ be the tangent [cotangent] bundle of M.

Definition 1: A Lorentzian manifold is a pair (M,g), *where* $g\in\sec T^{2,0}M$ *is a Lorentzian metric of signature* (1,3), *i.e., for all* $x\in M$, $T_xM\simeq T_x^*M\simeq\mathbb{R}^{1,3}$, *where* $\mathbb{R}^{1,3}$ *is the vector Minkowski space*.

Definition 2: A spacetime $\mathfrak{M}$ *is a pentuple* $(M,g,\boldsymbol{\nabla},\tau_g,\uparrow)$ *where* $(M,g,\tau_g,\uparrow)$ *is an oriented Lorentzian manifold (oriented by* τ_g*) and time oriented by an appropriated equivalence relation (denoted* $\uparrow$*) for the timelike vectors at the tangent space* T_xM, $\forall x\in M$. *(See Ref. 35 for details.)* $\boldsymbol{\nabla}$ *is a linear connection for* M *such that* $\boldsymbol{\nabla}g=0$.

Definition 3: Let $\mathbf{T}$ *and* $\mathbf{R}$ *be respectively, the torsion and curvature tensors of* $\boldsymbol{\nabla}$. *If in addition to the requirements of the previous definitions,* $\mathbf{T}(\boldsymbol{\nabla})=0$, *then* $\mathfrak{M}$ *is said to be a Lorentzian spacetime. The particular Lorentzian spacetime where* $M\simeq\mathbb{R}^4$ *and such that* $\mathbf{R}(\boldsymbol{\nabla})=0$ *is called Minkowski spacetime and will be denoted by* $\mathcal{M}$. *When* $\mathbf{T}(\boldsymbol{\nabla})$ is possibly nonzero, $\mathfrak{M}$ *is said to be a Riemann–Cartan spacetime (RCST). A particular RCST such that* $\mathbf{R}(\boldsymbol{\nabla})=0$ *is called a teleparallel spacetime*.

In what follows $P_{\mathrm{SO}_{1,3}^e}(M)$ denotes the principal bundle of oriented *Lorentz tetrads*. [We assume that the reader is acquainted with the structure of $P_{\mathrm{SO}_{1,3}^e}(M)$, whose sections are the time oriented and oriented orthonormal frames, each one associated by a local trivialization to a *unique* element of $SO_{1,3}^e(M)$. See, e.g., Refs. 16, 22, 29, 30.]

It is well known[32] that the natural operations on metric vector spaces, such as direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metrics. We have the following definition.

Definition 4: The Clifford bundle of the Lorentzian manifold (M,g) *is the bundle of algebras*,

$$\mathcal{C}\ell(M,g)=\bigcup_{x\in M}\mathcal{C}\ell(T_xM,g_x), \tag{1}$$

where $\mathcal{C}\ell(T_xM,g_x)$ *is the Clifford algebra associated with* (T_xM,g_x) *(see, e.g. Ref. 34).*

As is well known,[4,5,10] $\mathcal{C}\ell(M,g)$ is a *quotient* (or *factor*) bundle, namely

$$\mathcal{C}\ell(M,g)=\frac{\tau M}{\mathcal{J}(M,g)}, \tag{2}$$

where $\tau M=\oplus_{r=0}^{\infty}T^{0,r}M$ and $T^{(0,r)}M$ is the space of r-contravariant tensor fields, and $\mathcal{J}(M,g)$ is the bundle of ideals whose fibers are the two-sided ideals in τM generated by the elements of the form $a\otimes b+b\otimes a-2g(a,b)$, with $a,b\in TM$. In what follows, we denote the real Clifford algebra associated to $\mathbb{R}^{p,q}$ by $\mathbb{R}_{p,q}$. The even subalgebra of $\mathbb{R}_{p,q}$ will be denoted by $\mathbb{R}_{p,q}^0$ (see, e.g., Ref. 34).

Let $\pi_c:\mathcal{C}\ell(M,g)\to M$ be the canonical projection of $\mathcal{C}\ell(M,g)$ and let $\{U_\alpha\}$ be an open covering of M. There are trivialization mappings $\psi_i:\pi_c^{-1}(U_i)\to U_i\times\mathbb{R}_{1,3}$ of the form $\psi_i(p)=(\pi_c(p),\psi_{i,x}(p))=(x,\psi_{i,x}(p))$. If $x\in U_i\cap U_j$ and $p\in\pi_c^{-1}(x)$, then

$$\psi_{i,x}(p)=h_{ij}(x)\psi_{j,x}(p), \tag{3}$$

for $h_{ij}(x)\in\mathrm{Aut}(\mathbb{R}_{1,3})$, where $h_{ij}:U_i\cap U_j\to\mathrm{Aut}(\mathbb{R}_{1,3})$ are the transition mappings of $\mathcal{C}\ell(M,g)$. We know that every automorphism of $\mathbb{R}_{1,3}$ is *inner* and it follows that

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$$h_{ij}(x)\psi_{j,x}(p)=g_{ij}(x)\psi_{i,x}(p)g_{ij}(x)^{-1}, \tag{4}$$

for some $g_{ij}(x)\in\mathbb{R}_{1,3}^{\star}$, the group of invertible elements of $\mathbb{R}_{1,3}$.

Now, the group $\mathrm{SO}_{1,3}^{e}$ has as it is well known (see, e.g., Refs. 2, 3, 5, 21, 34) a natural extension in the Clifford algebra $\mathbb{R}_{1,3}$. Indeed we know that $\mathbb{R}_{1,3}^{\star}$ acts naturally on $\mathbb{R}_{1,3}$ as an algebra automorphism through its adjoint representation. A set of *lifts* of the transition functions of $\mathcal{C}\ell(M,g)$ is a set of $\mathbb{R}_{1,3}^{\star}$-valued functions $\{g_{ij}\}$ such that if

$$\begin{aligned}&\mathrm{Ad}:g\mapsto\mathrm{Ad}_g,\\&\mathrm{Ad}_g(a)=gag^{-1},\forall a\in\mathbb{R}_{1,3},\end{aligned}\tag{5}$$

then $\mathrm{Ad}_{g_{ij}}=h_{ij}$ in all intersections.

Also $\sigma=\mathrm{Ad}|_{\mathrm{Spin}_{1,3}^{e}}$ defines a group homeomorphism $\sigma:\mathrm{Spin}_{1,3}^{e}\to\mathrm{SO}_{1,3}^{e}$ which is onto with kernel $\mathbb{Z}_2$. [Recall that $\mathrm{Spin}_{1,3}^{e}=\{a\in\mathbb{R}_{1,3}^{0}:a\tilde{a}=1\}\simeq\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of the restricted Lorentz group $\mathrm{SO}_{1,3}^{e}$. See, e.g., Ref. 34.] We have that $\mathrm{Ad}_{-1}=\text{identity}$, and so $\mathrm{Ad}:\mathrm{Spin}_{1,3}^{e}\to\mathrm{Aut}(\mathbb{R}_{1,3})$ descends to a representation of $\mathrm{SO}_{1,3}^{e}$. Let us call Ad' this representation, i.e., $\mathrm{Ad}':\mathrm{SO}_{1,3}^{e}\to\mathrm{Aut}(\mathbb{R}_{1,3})$. Then we can write $\mathrm{Ad}'_{\sigma(g)}\,a=\mathrm{Ad}_g\,a=gag^{-1}$.

From this it is clear that the structure group of the Clifford bundle $\mathcal{C}\ell(M,g)$ is reducible from $\mathrm{Aut}(\mathbb{R}_{1,3})$ to $\mathrm{SO}_{1,3}^{e}$. This follows immediately from the Lorentzian structure of (M,g) and the fact that $\mathcal{C}\ell(M,g)$ is the exterior bundle where the fibers are equipped with the Clifford product. Thus the transition maps of the principal bundle of oriented Lorentz tetrads $P_{\mathrm{SO}_{1,3}^{e}}(M)$ can be (through Ad') taken as transition maps for the Clifford bundle. We then have[5]

$$\mathcal{C}\ell(M,g)=P_{\mathrm{SO}_{1,3}^{e}}(M)\times_{\mathrm{Ad}'}\mathbb{R}_{1,3}, \tag{6}$$

i.e., the Clifford bundle is an associated vector bundle to the principal bundle $P_{\mathrm{SO}_{1,3}^{e}}(M)$ of orthonormal Lorentz frames.

Definition 5: Sections of $\mathcal{C}\ell(M,g)$ *are called Clifford fields. (We note that the term Clifford fields was used in Ref. 34 for mappings from Minkowski spacetime to the Clifford algebra* $\mathbb{R}_{1,3}$*.)*

B. Spinor bundles

Definition 6: A spin structure on M *consists of a principal fiber bundle* $\pi_s:P_{\mathrm{Spin}_{1,3}^{e}}(M)\to M$ *(called the Spin Frame Bundle) with group* $\mathrm{Spin}_{1,3}^{e}$ *and a map*

$$s:P_{\mathrm{Spin}_{1,3}^{e}}(M)\to P_{\mathrm{SO}_{1,3}^{e}}(M), \tag{7}$$

satisfying the following conditions.

(i) $\pi(s(p))=\pi_s(p)\forall p\in P_{\mathrm{Spin}_{1,3}^{e}}(M)$; π is the projection map of the bundle $P_{\mathrm{SO}_{1,3}^{e}}(M)$.

(ii) $s(pu)=s(p)Ad_u$, $\forall p\in P_{\mathrm{Spin}_{1,3}^{e}}(M)$ and $Ad:\mathrm{Spin}_{1,3}^{e}\to\mathrm{Aut}(\mathbb{R}_{1,3})$, $Ad_u:\mathbb{R}_{1,3}\ni x\mapsto uxu^{-1}\in\mathbb{R}_{1,3}$.

Recall that minimal left (right) ideals of $\mathbb{R}_{p,q}$ are left (right) modules for $\mathbb{R}_{p,q}$.[34] In Ref. 34, covariant, algebraic and Dirac–Hestenes spinors [when $(p,q)=(1,3)$] were defined as certain equivalence classes in appropriate sets, and a *preliminary* definition for fields of these objects living on *Minkowski* spacetime was given. We are now interested in defining algebraic Dirac spinor fields and also Dirac–Hestenes spinor fields, on a general Riemann–Cartan spacetime (Definition 3), as sections of appropriate vector bundles (spinor bundles) associated to $P_{\mathrm{Spin}_{1,3}^{e}}(M)$. The compatibility between $P_{\mathrm{Spin}_{1,3}^{e}}(M)$ and $P_{\mathrm{SO}_{1,3}^{e}}(M)$, as captured in Definition 6, is essential for that matter.

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It is therefore natural to ask the following: When does a spin strucute exist on an oriented manifold M? The answer, which is a classical result (Refs. 1, 4, 5, 10, 15, 22, 29–31, 33, 32) is that the necessary and sufficient conditions for the existence of a spin structure on M is that the second Stiefel–Whitney class $w_2(M)$ of M is trivial. Moreover, when a spin structure exists, one can show that it is unique (modulo isomorfisms) if and only if $H^1(M,\mathbb{Z}_2)$ is trivial.

Remark 7: For a spacetime $\mathfrak{M}$ *(Definition 2), a spin structure exists if and only if* $P_{\mathrm{SO}^e_{1,3}}(M)$ *is a* trivial *bundle. This was originally shown by Geroch.*[16]

Definition 8: We call global sections $\xi \in \sec P_{\mathrm{SO}^e_{1,3}}(M)$ Lorentz frames *and global sections* $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ spin frames.

Remark 9: Recall that a principal bundle is trivial if and only if it admits a global section. Therefore, Geroch's result says that a (noncompact) spacetime admits a spin structure if and only if it admits a (globally defined) Lorentz frame. In fact, it is possible to replace $P_{\mathrm{SO}^e_{1,3}}(M)$ *by* $P_{\mathrm{Spin}^e_{1,3}}(M)$ *in Remark 7 (see Ref. 16, Footnote 25). In this way, when a (noncompact) spacetime admits a spin structure, the bundle* $P_{\mathrm{Spin}^e_{1,3}}(M)$ *is trivial and, therefore, every bundle associated to it is also trivial.*

Definition 10: An oriented manifold endowed with a spin structure will be called a spin manifold.

We now present the most usual definitions of spinor bundles appearing in the literature and next we find appropriate vector bundles such that particular sections are LIASF or DHSF. [We recall that there are some other (equivalent) definitions of spinor bundles that we are not going to introduce in this paper as, e.g., the one given in Ref. 6 in terms of mappings from $P_{\mathrm{Spin}^e_{1,3}}$ to some appropriate vector space.]

Definition 11: A real spinor bundle for M *is a vector bundle,*

$$S(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_l} \mathbf{M}, \tag{8}$$

where $\mathbf{M}$ *is a left module for* $\mathbb{R}_{1,3}$ *and* μ_l *is a representation of* $\mathrm{Spin}^e_{1,3}$ *on* $End(\mathbf{M})$ *given by left multiplication by elements of* $\mathrm{Spin}^e_{1,3}$.

Definition 12: The dual bundle $S^\star(M)$ *is a real spinor bundle,*

$$S^\star(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_r} \mathbf{M}^\star, \tag{9}$$

where $\mathbf{M}^\star$ *is a right module for* $\mathbb{R}_{1,3}$ *and* μ_r *is a representation* of $\mathrm{Spin}^e_{1,3}$ *in* $End(\mathbf{M})$ *given by right multiplication by (inverse) elements of* $\mathrm{Spin}^e_{1,3}$. *[More precisely, this means that given* $u \in \mathrm{Spin}^e_{1,3}$, $a \in \mathbf{M}^*$, $\mu_r(u)a = au^{-1}$, *so that* $\mu_r(uu')a = a(uu')^{-1} = au'^{-1}u^{-1} = \mu_r(u)\mu_r(u')a$.*]*

Definition 13: A complex spinor bundle for M *is a vector bundle,*

$$S_c(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathbf{M}_c, \tag{10}$$

where $\mathbf{M}_c$ *is a complex left module for* $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, *and where* μ_c *is a representation of* $\mathrm{Spin}^e_{1,3}$ *in* $\mathrm{End}(\mathbf{M}_c)$ *given by left multiplication by elements of* $\mathrm{Spin}^e_{1,3}$.

Definition 14: The dual complex spinor bundle for M *is a vector bundle,*

$$S^\star_c(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_c} \mathbf{M}^\star_c, \tag{11}$$

where $\mathbf{M}^\star_c$ *is a complex right module for* $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, *and where* μ_c *is a representation of* $\mathrm{Spin}^e_{1,3}$ *in* $End(\mathbf{M}_c)$ *given by right multiplication by (inverse) elements of* $\mathrm{Spin}^e_{1,3}$. *[More precisely, this means that given* $u \in \mathrm{Spin}^e_{1,3}$, $a \in \mathbf{M}^*$, $\mu_r(u)a = au^{-1}$.*]*

Taking, e.g., $\mathbf{M}_c = \mathbb{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathrm{Spin}^e_{1,3} \cong \mathrm{SL}(2,\mathbb{C})$ in $\mathrm{End}(\mathbb{C}^4)$, we immediately recognize the usual definition of the covariant spinor bundle of M as given, e.g., in (Refs. 7, 8, 9, 15, 29, 30).

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C. Left spin-Clifford bundle

As shown in Ref. 34, besides the ideal $I=\mathbb{R}_{1,3}\frac{1}{2}(1+E_0)$, other ideals exist in $\mathbb{R}_{1,3}$ that are only *algebraically* equivalent to this one. (This fact gives rise to a large class of multivector Dirac equations in flat spacetime, generalizing the Dirac–Hestenes equation.[23,24]) In order to capture all possibilities we recall that $\mathbb{R}_{1,3}$ can be considered as a module over itself by left (or right) multiplication. We are thus led to the following definition.

Definition 15: The left real spin-Clifford bundle of M is the vector bundle,

$$\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)=P_{\mathrm{Spin}^e_{1,3}}(M)\times_l\mathbb{R}_{1,3}, \tag{12}$$

where l is the representation of $\mathrm{Spin}^e_{1,3}$ *on* $\mathbb{R}_{1,3}$ *given by* $l(a)x=ax$. *Sections of* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *are called left spin-Clifford fields.*

Remark 16: $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *is a "principal* $\mathbb{R}_{1,3}$*-bundle," i.e., it admits a free action of* $\mathbb{R}_{1,3}$ *on the right,[5] which is denoted by* R_g, $g\in\mathbb{R}_{1,3}$. *This will be considered in Sec. V.*

Remark 17: There is a natural *embedding* $P_{\mathrm{Spin}^e_{1,3}}(M)\hookrightarrow\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *which comes from the embedding* $\mathrm{Spin}^e_{1,3}\hookrightarrow\mathbb{R}^0_{1,3}$. *(The symbol $A\hookrightarrow B$ means that A is embedded in B and $A\subseteq B$.) Hence (as we shall see in more details below), every real left spinor bundle (see Definition 15) for M can be captured from* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$, *which is a vector bundle very different from* $\mathcal{C}\ell(M,g)$. *Their relation is presented below, but before that we give the following definition.*

Definition 18: Let $I(M)$ be a subbundle of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *such that there exists a primitive idempotent* $\mathbf{e}$ *of* $\mathbb{R}_{1,3}$ *(see, e.g., Ref. 34) with*

$$R_{\mathbf{e}}\mathbf{\Psi}=\mathbf{\Psi}\mathbf{e}=\mathbf{\Psi}, \tag{13}$$

for all $\Psi\in\sec I(M)\subset\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$. *Then, $I(M)$ is called a subbundle of left ideal algebraic spinor fields. Any* $\Psi\in\sec I(M)\subset\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *is called a left ideal algebraic spinor field (LIASF). $I(M)$ can be thought of as a real spinor bundle for M such that* $\mathbf{M}$ *in Eq. (8) is a minimal left ideal of* $\mathbb{R}_{1,3}$.

Definition 19: Two subbundles $I(M)$ and $I'(M)$ of LIASF are said to be geometrically equivalent if the idempotents e, $e'\in\mathbb{R}_{1,3}$ (appearing in the previous definition) are related by an element $u\in\mathrm{Spin}^e_{1,3}$, *i.e.,* $e'=ueu^{-1}$.

Definition 20: The right *real spin-Clifford bundle of M is the vector bundle*

$$\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)=P_{\mathrm{Spin}^e_{1,3}}(M)\times_r\mathbb{R}_{1,3}. \tag{14}$$

Sections of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$ *are called right spin-Clifford fields.*

In Eq. (14) r refers to the representation of $\mathrm{Spin}^e_{1,3}$ on $\mathbb{R}_{1,3}$ given by $r(a)x=xa^{-1}$. As in the case for the left real spin-Clifford bundle, there is a natural embedding $P_{\mathrm{Spin}^e_{1,3}}(M)\hookrightarrow\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$ which comes from the embedding $\mathrm{Spin}^e_{1,3}\hookrightarrow\mathbb{R}^0_{1,3}$. There exists also a natural left action L_a of $a\in\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$. This will be proved in Sec. V.

Definition 21: Let $I^\star(M)$ be a subbundle of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$ *such that there exists a primitive idempotent element* $\mathbf{e}$ *of* $\mathbb{R}_{1,3}$ *with*

$$L_{\mathbf{e}}\Psi=e\Psi=\Psi, \tag{15}$$

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for any $\Psi \in \sec I^{\star}(M) \subset \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$. *Then,* $I^{\star}(M)$ *is called a subundle of right ideal algebraic spinor fields. Any* $\Psi \in \sec I^{\star}(M) \subset \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$ *is called a right ideal algebraic spinor field (RIASF).* $I^{\star}(M)$ *can be thought of as a real spinor bundle for* M *such that* $\mathbf{M}^{\star}$ *in Eq. (9) is a minimal right ideal of* $\mathbb{R}_{1,3}$.

Definition 22: Two subbundles $I^{*}(M)$ *and* $I^{*\prime}(M)$ *of RIASF are said to be geometrically equivalent if the idempotents* e, $e' \in \mathbb{R}_{1,3}$ *(appearing in the previous definition) are related by an element* $u \in \mathrm{Spin}^e_{1,3}$, *i.e.,* $e' = ueu^{-1}$.

Proposition 23: In a spin manifold, we have

$$\mathcal{C}\ell(M,g) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}.$$

Proof: Remember once again that the representation,

$$\mathrm{Ad}: \mathrm{Spin}^e_{1,3} \to \mathrm{Aut}(\mathbb{R}_{1,3}), \quad \mathrm{Ad}_u a = uau^{-1}, \quad u \in \mathrm{Spin}^e_{1,3},$$

is such that $\mathrm{Ad}_{-1} = \mathrm{identity}$ and so Ad descends to a representation Ad′ of $\mathrm{SO}^e_{1,3}$ which we considered above. It follows that when $P_{\mathrm{Spin}^e_{1,3}}(M)$ exists $\mathcal{C}\ell(M,g) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$. ■

D. Bundle of modules over a bundle of algebras

Proposition 24: $S(M)$ [*or* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$] *is a bundle of (left)* modules *over the bundle of algebras* $\mathcal{C}\ell(M,g)$. *In particular, the sections of the spinor bundle S(M) [or* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}$*(M)] constitute a module over the sections of the Clifford bundle.*

For the proof, see Ref. 5, p. 97.

Corollary 25: Let Φ, $\Psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *and* $\Psi \neq 0$. *Then there exists* $\psi \in \sec \mathcal{C}\ell(M,g)$ *such that*

$$\Psi = \psi\Phi. \tag{16}$$

Proof: It is an immediate consequence of Proposition 24. ■

So, the corollary allows us to identify a *correspondence* between some sections of $\mathcal{C}\ell(M,g)$ and some sections of $I(M)$ or $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ once we fix a section on $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$. This and other correspondences will be essential for the theory of Sec. V. Once we clarified the meaning of a bundle of modules $S(M)$ over a bundle of algebras $\mathcal{C}\ell(M,g)$, we can give the following.

Definition 26: Two real left spinor bundles (see Definition 15) are equivalent if and only if they are equivalent as bundles of $\mathcal{C}\ell(M,g)$ *modules.*

Remark 27: Of course, geometrically equivalenty real left spinor bundles are equivalent.

Remark 28: In what follows we denote the complexified left spin Clifford bundle by $\mathbb{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{l} \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\mathrm{Spin}^e_{1,3}}(M) \times_{r} \mathbb{R}_{4,1}$ *and the complexified right spin Clifford bundle by* $\mathbb{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{r} \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\mathrm{Spin}^e_{1,3}}(M) \times_{r} \mathbb{R}_{4,1}$.

III. DIRAC–HESTENES SPINOR FIELDS

Let $\mathbf{E}^{\mu}$, $\mu = 0,1,2,3$ be the canonical basis of $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ which generates the algebra $\mathbb{R}_{1,3}$. They satisfy the basic relation $\mathbf{E}^{\mu}\mathbf{E}^{\nu} + \mathbf{E}^{\nu}\mathbf{E}^{\mu} = 2\eta^{\mu\nu}$. As shown, e.g., in Ref. 34,

$$\mathbf{e} = \frac{1}{2}(1 + \mathbf{E}^{0}) \in \mathbb{R}_{1,3}, \tag{17}$$

is a primitive idempotent of $\mathbb{R}_{1,3}$ and

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$$\mathbf{f}=\frac{1}{2}(1+\mathbf{E}^0)\frac{1}{2}(1+i\mathbf{E}^2\mathbf{E}^1)\in\mathbb{C}\otimes\mathbb{R}_{1,3} \tag{18}$$

is a primitive idempotent of $\mathbb{C}\otimes\mathbb{R}_{1,3}$. Now, let $\mathbf{I}=\mathbb{R}_{1,3}\mathbf{e}$ and $\mathbf{I}_{\mathbb{C}}=\mathbb{C}\otimes\mathbb{R}_{1,3}\mathbf{f}$ be, respectively, the minimal left ideals of $\mathbb{R}_{1,3}$ and $\mathbb{C}\otimes\mathbb{R}_{1,3}$ generated by $\mathbf{e}$ and $\mathbf{f}$. Let $\phi=\phi\mathbf{e}\in\mathbf{I}$ and $\Psi=\Psi\mathbf{f}\in\mathbf{I}_{\mathbb{C}}$. Then, any $\phi\in\mathbf{I}$ can be written as

$$\phi=\psi\mathbf{e}, \tag{19}$$

with $\psi\in\mathbb{R}_{1,3}^0$. Analogously, any $\Psi\in\mathbf{I}_{\mathbb{C}}$ can be written as

$$\Psi=\psi\mathbf{e}\frac{1}{2}(1+i\mathbf{E}^2\mathbf{E}^1), \tag{20}$$

with $\psi\in\mathbb{R}_{1,3}^0$.

Now, $\mathbb{C}\otimes\mathbb{R}_{1,3}\simeq\mathbb{R}_{4,1}\simeq\mathbb{C}(4)$, where $\mathbb{C}(4)$ is the algebra of the 4×4 complex matrices. We can verify that

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

is a primitive idempotent of $\mathbb{C}(4)$ which is a matrix representation of $\mathbf{f}$. In this way we can prove (as shown, e.g., in Ref. 34) that there is a bijection between column spinors, i.e., elements of $\mathbb{C}^4$ (the complex 4-dimensional vector space) and the elements of $\mathbf{I}_{\mathbb{C}}$. All that, plus the definitions of the left real and complex spin bundles and the subbundle $I(M)$, suggests the following.

Definition 29: Let $\Phi\in\sec I(M)\subset\sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ *be as in Definition 18, i.e.,*

$$R_{\mathbf{e}}\Phi=\Phi\mathbf{e}=\Phi,\quad \mathbf{e}^2=\mathbf{e}=\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{E}^{\mathbf{0}})\in\mathbb{R}_{1,3}. \tag{22}$$

A Dirac–Hestenes Spinor field (DHSF) associated with Φ *is an even section* ψ *of* $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ *such that*

$$\Phi=\psi\mathbf{e}. \tag{23}$$

[Note that it is meaningful to speak about even (or odd) elements in $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ since $\mathrm{Spin}^e_{1,3}\subseteq\mathbb{R}^0_{1,3}$.]

Remark 30: An equivalent definition of a DHSF is the following. Let $\Psi\in\sec\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ *be such that*

$$R_{\mathbf{f}}\Psi=\Psi\mathbf{f}=\Psi,\quad \mathbf{f}^2=\mathbf{f}=\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{E}^{\mathbf{0}})\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{i}\mathbf{E}^{\mathbf{2}}\mathbf{E}^{\mathbf{1}})\in\mathbb{C}\otimes\mathbb{R}_{1,3}. \tag{24}$$

Then, a DHSF associated to Ψ *is an even section* ψ *of* $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)\subset\mathbb{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ *such that*

$$\Psi=\psi\mathbf{f}. \tag{25}$$

Remark 31: In what follows, when we refer to a Dirac–Hestenes spinor field ψ *we omit for simplicity the wording associated with* Φ (*or* Ψ). *It is very important to observe that DHSF are not sums of even multivector (tensor) fields although, under a local trivialization,* ψ

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$\in \sec C\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *is mapped on an even element of* $\mathbb{R}_{1,3}$. *We emphasize that DHSF are particular sections of a spinor bundle, not of the Clifford bundle. However, we show in Sec. V how these objects have representatives in the Clifford bundle.*

IV. THE MANY FACES OF THE DIRAC EQUATION

A. Dirac equation for covariant Dirac fields

As is well known,[8] a *covariant* Dirac spinor field is a section $\Psi \in \sec S_c(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mu_l} \mathbb{C}^4$. Let $(U = M, \Phi), \Phi(\boldsymbol{\Psi}) = (x, |\Psi(x)\rangle)$ be a global trivialization corresponding to a spin frame Ξ (Definition 8), such that

$$s(\Xi) = \{e_a\} \in P_{\mathrm{SO}^e_{1,3}}(M), \quad e^a \in \sec C\ell(M,g),$$
$$e^a e^b + e^b e^a = 2\,\eta^{ab}, a,b = 0,1,2,3 \tag{26}$$

(see Definition 6). The usual Dirac equation in a Lorentzian spacetime for the spinor field Ψ—in interaction with an electromagnetic field $A \in \sec \Lambda^1(M) \subset \sec \mathcal{C}\ell(M,g)$—is then[11]

$$i\,\gamma^a(\boldsymbol{\nabla}^s_{e_a} + iqA_a)|\Psi(x)\rangle - m|\Psi(x)\rangle = 0, \tag{27}$$

where $\gamma^a \in \mathbb{C}(4)$, $a = 0,1,2,3$ is a set of *constant* Dirac *matrices* satisfying

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\,\eta^{ab}. \tag{28}$$

[We denote the space of sections of p-vectors by $\sec \Lambda^p(M)$.]

B. Dirac equation in $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$

Due to the one-to-one correspondence between *ideal* sections of $\mathbb{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$, $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ and of $S_c(M)$ as explained in Sec. III, we can *translate* the Dirac equation (27) for a covariant spinor field into an equation for a spinor field, which is a section of $\mathbb{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$, and finally write an equivalent equation for a DHSF $\psi \in \sec C\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$. In order to do that we introduce the spin-Dirac operator.

Definition 32: The (spin) Dirac operator acting on sections of $C\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *is the first order differential operator,*[5]

$$D^s = e^a \boldsymbol{\nabla}^s_{e_a}, \tag{29}$$

where $\{e^a\}$ *is as in Eq. (26) and* ∇^s *is the spinor covariant derivative (see the Appendix).*

Now we give the details of the inverse translation. We start with the following equation which we call Dirac equation in $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$, denoted DE$\mathcal{C}\ell^l$:

$$D^s\psi\mathbf{E}^{21} - m\psi\mathbf{E}^0 - qA\boldsymbol{\psi} = 0, \tag{30}$$

where $\psi \in \sec \mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ is a DHSF and the $\mathbf{E}^a \in \mathbb{R}_{1,3}$ are such that $\mathbf{E}^a\mathbf{E}^b + \mathbf{E}^b\mathbf{E}^a = 2\,\eta^{ab}$. Multiplying Eq. (30) on the right by the idempotent $\mathbf{f} = \frac{1}{2}(1+\mathbf{E}^0)\frac{1}{2}(1+i\mathbf{E}^2\mathbf{E}^1) \in \mathbb{C} \otimes \mathbb{R}_{1,3}$ we get after some simple algebraic manipulations the following equation for the (complex) left ideal spin-Clifford field $\Psi = \psi\mathbf{f} \in \sec \mathbb{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$:

$$iD^s\Psi - m\Psi - qA\Psi = 0. \tag{31}$$

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Now we can easily show, using the methods of Ref. 34, that given any global trivializations $(U=M,\Theta)$ and $(U=M,\Phi)$, of $\mathcal{C}\ell(M,g)$ and $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M)$, there exists matrix representations of the $\{e^{a}\}$ that are equal to the Dirac matrices $\boldsymbol{\gamma}^{a}$ [appearing in Eq. (27)]. In that way the correspondence between Eqs. (27), (30) and (31) is proved.

Remark 33: We emphasize at this point that we call Eq. (30) the DE$\mathcal{C}\ell^{l}$. *It looks similar to the Dirac–Hestenes equation (on Minkowski spacetime) discussed in Ref. 34, but it is indeed very different regarding its mathematical nature. It is an intrinsic equation satisfied by a legitimate spinor field, namely a DHSF* $\psi\in\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M)$. *The question naturally arises: May we write an equation with the same mathematical information of Eq. (30) but satisfied by objects living on the Clifford bundle* $\mathcal{C}\ell(M,g)$ *of an arbitrary Lorentzian spacetime, admitting a spin structure? In the next section we show that the answer to that question is yes.*

C. Electromagnetic gauge invariance of the DE$\mathcal{C}\ell^{l}$

Proposition 34: The DE$\mathcal{C}\ell^{l}$ *is invariant under electromagnetic gauge transformations,*

$$\psi\mapsto\psi'=\psi e^{q\mathbf{E}^{21}\chi}, \tag{32}$$

$$A\mapsto A+\partial\chi, \tag{33}$$

$$\omega_{e_a}\mapsto\omega_{e_a}, \tag{34}$$

$$\psi,\psi'\in\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M), \tag{35}$$

$$A\in\sec\Lambda^{1}(M)\subset\sec\mathcal{C}\ell(M,g), \tag{36}$$

with ψ,ψ' *distinct DHSF, and where* $\chi:M\to\mathbb{R}\subset\mathbb{R}_{1,3}$ *is a gauge function.*

Proof: The proof is obtained by direct verification. ■

Remark 35: We note that, for the DE$\mathcal{C}\ell^{l}$, local rotations and electromagnetic gauge transformations are very different mathematical transformations, without any obvious geometrical link between them, differently of what seems to be the case for the Dirac–Hestenes equation, which is studied in the next section.

V. THE DIRAC–HESTENES EQUATION (DHE)

We obtained above a Dirac equation, which we called DE$C\ell^{l}$, describing the motion of spinor fields represented by sections Ψ of $C\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ in interaction with an electromagnetic field $A\in\sec C\ell(M,g)$,

$$D^{s}\Psi\mathbf{E}^{21}-qA\Psi=m\Psi\mathbf{E}^{0}, \tag{37}$$

where $D^{s}=e^{a}\boldsymbol{\nabla}^{s}_{e_a}$, $\{e^{a}\}$ is given by Eq. (26), $\boldsymbol{\nabla}^{s}_{e_a}$ is the natural spinor covariant derivative acting on $\sec C\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$ (see the Appendix), and $\{\mathbf{E}^{a}\}\in\mathbb{R}^{1,3}\subseteq\mathbb{R}_{1,3}$ is such that $\mathbf{E}^{a}\mathbf{E}^{b}+\mathbf{E}^{b}\mathbf{E}^{a}=2\eta^{ab}$. As we already mentioned, although Eq. (37) is written in a kind of Clifford bundle [i.e. $C\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$], it does not suffer from the inconsistency of representing spinors as pure differential forms and, in fact, the object Ψ behaves as it should under Lorentz transformations.

As a matter of fact, Eq. (37) can be thought of as a mere *rewriting* of the usual Dirac equation, where the role of the constant gamma matrices is undertaken by the constant elements $\{\mathbf{E}^{a}\}$ in $\mathbb{R}_{1,3}$ and by the set $\{e^{a}\}$. In this way, Eq. (37) is *not* a kind of Dirac–Hestenes equation as discussed, e.g., in Ref. 34. It suffices to say that (i) the state of the electron, represented by Ψ, is not a *Clifford field* and (ii) the $\mathbf{E}^{a}$'s are just *constant* elements of $\mathbb{R}_{1,3}$ and not sections of vectors in $\mathcal{C}\ell(M,g)$. Nevertheless, as we show in the following, Eq. (37) does lead to a multivector Dirac

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equation once we carefully employ the theory of right and left actions on the various Clifford bundles introduced earlier. It is the multivector equation to be derived below that we call the DHE (of course, we can write an equivalent multiform equation). We shall need several preliminary results that we collect in the next two subsections.

A. The various natural actions on the vector bundles associated to $P_{\mathrm{Spin}^e_{1,3}}(M)$

Recall that, when M is a spin manifold the following occurs.

(i) The elements of $\mathcal{C}\ell(M,g)=P_{\mathrm{Spin}^e_{1,3}}(M)\times_{\mathrm{Ad}}\mathbb{R}_{1,3}$ are equivalence classes $[(p,a)]$ of pairs (p,a), where $p\in P_{\mathrm{Spin}^e_{1,3}}(M)$, $a\in\mathbb{R}_{1,3}$ and $(p,a)\sim(p',a')\Leftrightarrow p'=pu^{-1}$, $a'=uau^{-1}$, for some $u\in\mathrm{Spin}^e_{1,3}$.

(ii) The elements of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ are equivalence classes of pairs (p,a), where $p\in P_{\mathrm{Spin}^e_{1,3}}(M)$, $a\in\mathbb{R}_{1,3}$ and $(p,a)\sim(p',a')\Leftrightarrow p'=pu^{-1}$, $a'=ua$, for some $u\in\mathrm{Spin}^e_{1,3}$.

(iii) The elements of $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$ are equivalence classes of pairs (p,a), where $p\in P_{\mathrm{Spin}^e_{1,3}}(M)$, $a\in\mathbb{R}_{1,3}$ and $(p,a)\sim(p',a')\Leftrightarrow p'=pu^{-1}$, $a'=au^{-1}$, for some $u\in\mathrm{Spin}^e_{1,3}$.

In this way, it is possible to define the following natural actions on these associated bundles.

Proposition 36: There is a natural right action of $\mathbb{R}_{1,3}$ *on* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *and a natural left action of* $\mathbb{R}_{1,3}$ *on* $\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,g)$.

Proof: Given $b\in\mathbb{R}_{1,3}$ and $\alpha\in\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$, select a representative (p,a) for α and define $\alpha b:=[(p,ab)]$. If another representative (pu^{-1},ua) is chosen for α, we have $(pu^{-1},uab)\sim(p,ab)$ and thus αb is a well-defined element of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$. ∎

Let us denote the space of $\mathbb{R}_{1,3}$-valued smooth functions on M by $\mathcal{F}(M,\mathbb{R}_{1,3})$. Then, the above proposition immediately yields the following.

Corollary 37: There is a natural right action of $\mathcal{F}(M,\mathbb{R}_{1,3})$ *on* $\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *and a natural left action of* $\mathcal{F}(M,\mathbb{R}_{1,3})$ *on* $\sec\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M,g)$.

Proposition 38: There is a natural left action of $\sec\mathcal{C}\ell(M,g)$ *on* $\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ *and a natural right action of* $\sec\mathcal{C}\ell(M,g)$ *on* $\sec\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$.

Proof: Given $\alpha\in\sec\mathcal{C}\ell(M,g)$ and $\beta\in\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$, select representatives (p,a) for $\alpha(x)$ and (p,b) for $\beta(x)$ [with $p\in\pi^{-1}(x)$] and define $(\alpha\beta)(x):=[(p,ab)]\in\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$. If alternative representatives (pu^{-1},uau^{-1}) and (pu^{-1},ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have

$$(pu^{-1},uau^{-1}ub)=(pu^{-1},uab)\sim(p,ab),$$

and thus $(\alpha\beta)(x)$ is a well-defined element of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M,g)$. ∎

Proposition 39: There is a natural pairing,

$$\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)\times\sec\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)\to\sec\mathcal{C}\ell(M,g).$$

Proof: Given $\alpha\in\sec\mathcal{C}\ell^{l}_{\mathrm{Spin}^e_{1,3}}(M)$ and $\beta\in\sec\mathcal{C}\ell^{r}_{\mathrm{Spin}^e_{1,3}}(M)$, select representatives (p,a) for $\alpha(x)$ and (p,b) for $\beta(x)$ [with $p\in\pi^{-1}(x)$] and define $(\alpha\beta)(x):=[(p,ab)]\in\mathcal{C}\ell(M,g)$. If alternative representatives (pu^{-1},ua) and (pu^{-1},bu^{-1}) are chosen for $\alpha(x)$ and $\beta(x)$, we have $(pu^{-1},uabu^{-1})\sim(p,ab)$ and thus $(\alpha\beta)(x)$ is a well-defined element of $C\ell(M,g)$. ∎

Proposition 40: There is a natural pairing,

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$$\sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M) \times \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M) \to \mathcal{F}(M,\mathbb{R}_{1,3}).$$

Proof: Given $\alpha \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ and $\beta \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$, select representatives (p,a) for $\alpha(x)$ and (p,b) for $\beta(x)$ [with $p \in \pi^{-1}(x)$] and define $(\alpha\beta)(x) := ab \in \mathbb{R}_{1,3}$. If alternative representatives (pu^{-1}, au^{-1}) and (pu^{-1}, ub) are chosen for $\alpha(x)$ and $\beta(x)$, we have $au^{-1}ub = ab$ and thus $(\alpha\beta)(x)$ is a well-defined element of $\mathbb{R}_{1,3}$. ■

B. Fiducial sections associated with a spin frame

We start by exploring the possibility of defining "unit sections" on the various vector bundles associated with the principal bundle $P_{\mathrm{Spin}^e_{1,3}}(M)$. It immediately follows from the definition given by Eq. (1) that the unit section $\mathbf{1} \in \sec \mathcal{C}\ell(M,g)$, given by $x \mapsto 1 \in \mathcal{C}\ell(T_xM, g_x)$, is certainly well defined. For future reference, let us consider how this can also be seen from the associated bundle structure of $P_{\mathrm{Spin}^e_{1,3}}(M) \times_{ad} \mathbb{R}_{1,3}$.

Let

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathrm{Spin}^e_{1,3}, \quad \Phi_j : \pi^{-1}(U_j) \to U_j \times \mathrm{Spin}^e_{1,3}$$

be two local trivializations for $P_{\mathrm{Spin}^e_{1,3}}(M)$, with

$$\Phi_i(u) = (\pi(u) = x, \phi_{i,x}(u)), \quad \Phi_j(u) = (\pi(u) = x, \phi_{j,x}(u)).$$

Recall that the transition function on $g_{ij} : U_i \cap U_j \to \mathrm{Spin}^e_{1,3}$ is then given by

$$g_{ij}(x) = \phi_{i,x}(u) \circ \phi_{j,x}(u)^{-1},$$

which does not depend on u.

Proposition 41: $\mathcal{C}\ell(M,g)$ has a naturally defined global unit section.

Proof: For the associated bundle $\mathcal{C}\ell(M,g) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$, the transition functions corresponding to local trivializations,

$$\Psi_i : \pi_c^{-1}(U_i) \to U_i \times \mathbb{R}_{1,3}, \quad \Psi_j : \pi_c^{-1}(U_j) \to U_j \times \mathbb{R}_{1,3}, \tag{38}$$

are given by $h_{ij}(x) = Ad_{g_{ij}(x)}$. Define the local sections

$$\mathbf{1}_i(x) = \Psi_i^{-1}(x,1), \quad \mathbf{1}_j(x) = \Psi_j^{-1}(x,1), \tag{39}$$

where 1 is the unit element of $\mathbb{R}_{1,3}$. Since $h_{ij}(x) \cdot 1 = \mathrm{Ad}_{g_{ij}(x)}(1) = g_{ij}(x) 1 g_{ij}(x)^{-1} = 1$, we see that the expressions above uniquely define a global section $\mathbf{1} \in \mathcal{C}\ell(M,g)$ with $\mathbf{1}|_{U_i} = \mathbf{1}_i$. ■

It is clear that such a result can be immediately generalized for the Clifford bundle $\mathcal{C}\ell_{p,q}(M,g)$, of any n-dimensional manifold endowed with a metric of arbitrary signature (p,q) (where $n = p+q$). Now, we observe also that the left (and also the right) spin-Clifford bundle can be generalized in an obvious way for any spin manifold of arbitrary finite dimension $n = p+q$, with a metric of arbitrary signature (p,q). However, another important difference between $\mathcal{C}\ell(M,g)$ and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M)$ or $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ is that these latter bundles only admit a global unit section if they are *trivial*.

Proposition 42: There exists an unit section on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M)$ [and also on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M)$] if and only if $P_{\mathrm{Spin}^e_{p,q}}(M)$ is trivial.

Proof: We show the necessity for the case of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M)$, the sufficiency is trivial. [The proof for the case of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M)$ is analogous.] For $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M)$, the transition functions corresponding to local trivializations,

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$$\Omega_i : \pi_{sc}^{-1}(U_i) \to U_i \times \mathbb{R}_{p,q}, \quad \Omega_j : \pi_{sc}^{-1}(U_j) \to U_j \times \mathbb{R}_{p,q}, \tag{40}$$

are given by $k_{ij}(x) = R_{g_{ij}(x)}$, with $R_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, x \mapsto xa^{-1}$. Let 1 be the unit element of $\mathbb{R}_{1,3}$. A unit section in $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{p,q}}(M)$—if it exists—is written in terms of these two local trivializations as

$$\mathbf{1}^r_i(x) = \Omega_i^{-1}(x,1), \quad \mathbf{1}^r_j(x) = \Omega_j^{-1}(x,1), \tag{41}$$

and we must have $\mathbf{1}^r_i(x) = \mathbf{1}^r_j(x)$, $\forall x \in U_i \cap U_j$. As $\Omega_i(\mathbf{1}^r_i(x)) = (x,1) = \Omega_j(\mathbf{1}^r_j(x))$, we have $\mathbf{1}^r_i(x) = \mathbf{1}^r_j(x) \Leftrightarrow 1 = k_{ij}(x) \cdot 1 \Leftrightarrow 1 = 1 g_{ij}(x)^{-1} \Leftrightarrow g_{ij}(x) = 1$. This proves the proposition. ∎

Remark 43: For general spin manifolds, the bundle $P_{\mathrm{Spin}^e_{p,q}}(M)$ is not necessarily trivial for arbitrary (p,q), but Geroch's theorem (Remark 9) warrants that, for the special case $(p,q) = (1,3)$ with M noncompact, $P_{\mathrm{Spin}^e_{1,3}}(M)$ is trivial. By the above proposition, we then see that $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ and also $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ have global "unit sections." It is most important to note, however, that each different choice of a (global) trivialization Ω_i on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ [respectively, $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{p,q}}(M)$] induces a different global unit section $\mathbf{1}^r_i$ (respectively, $\mathbf{1}^l_i$). Therefore, even in this case there is no canonical unit section on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ [respectively, on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$].

By Remark 9, when the (noncompact) spacetime M is a spin manifold, the bundle $P_{\mathrm{Spin}^e_{1,3}}(M)$ admits global sections. With this in mind, let us fix a spin frame Ξ for M. This induces a global trivialization for $P_{\mathrm{Spin}^e_{1,3}}(M)$, which we denote by $\Phi_\Xi : P_{\mathrm{Spin}^e_{1,3}}(M) \to M \times \mathrm{Spin}^e_{1,3}$, with $\Phi_\Xi^{-1}(x,1) = \Xi(x)$. As we show in the following, the spin frame Ξ can also be used to induce certain fiducial global sections on the various vector bundles associated with $P_{\mathrm{Spin}^e_{1,3}}(M)$.

(i) $\mathcal{C}\ell(M,g)$ Let $\{\mathbf{E}^a\}$ be a fixed orthonormal basis of $\mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$ (which can be thought of as the *canonical* basis of $\mathbb{R}^{1,3}$). We define basis sections in $\mathcal{C}\ell(M,g) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$ by $e_a(x) = [(\Xi(x), \mathbf{E}_a)]$. Of course, this induces a multivector basis $\{e_I(x)\}$ for each $x \in M$. Note that a more precise notation for e_a would be, for instance, $e_a^{(\Xi)}$.

(ii) $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ Let $\mathbf{1}^l_\Xi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ be defined by $\mathbf{1}^l_\Xi(x) = [(\Xi(x),1)]$. Then the natural right action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ leads to $\mathbf{1}^l_\Xi(x)a = [(\Xi(x),a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from Corollary 37 that an arbitrary section $\alpha \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ can be written as $\alpha = \mathbf{1}^l_\Xi f$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

(iii) $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ Let $\mathbf{1}^r_\Xi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,g)$ be defined by $\mathbf{1}^r_\Xi(x) = [(\Xi(x),1)]$. Then the natural left action of $\mathbb{R}_{1,3}$ on $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ leads to $a\mathbf{1}^r_\Xi(x) = [(\Xi(x),a)]$ for all $a \in \mathbb{R}_{1,3}$. It follows from Corollary 37 that an arbitrary section $\alpha \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ can be written as $\alpha = f\mathbf{1}^r_\Xi$, with $f \in \mathcal{F}(M, \mathbb{R}_{1,3})$.

Now recall (Definition 6) that a spin structure on M is a 2-1 bundle map $s : P_{\mathrm{Spin}^e_{1,3}}(M) \to P_{\mathrm{SO}^e_{1,3}}(M)$ such that $s(pu) = s(p)\mathrm{Ad}_u$, $\forall p \in P_{\mathrm{Spin}^e_{1,3}}(M)$, $u \in \mathrm{Spin}^e_{1,3}$, where $\mathrm{Ad} : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}$, $\mathrm{Ad}_u : x \mapsto uxu^{-1}$. We see that the specification of the global section in the case (i) above is compatible with the Lorentz frame $\{e_a\} = s(\Xi)$ assigned by s. More precisely, for each $x \in M$, the element $s(\Xi(x)) \in P_{\mathrm{SO}^e_{1,3}}(M)$ is to be regarded as a proper isometry $s(\Xi(x)) : \mathbb{R}^{1,3} \to T_xM$, so that $e_a(x) := s(p) \cdot \mathbf{E}_a$ yields a Lorentz frame $\{e_a\}$ on M, which we denoted by $s(\Xi)$. On the other hand, $\mathcal{C}\ell(M,g)$ is isomorphic to $P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}$, and we can always arrange things so that $e_a(x)$ is represented in this bundle as $e_a(x) = [(\Xi(x), \mathbf{E}_a)]$. In fact, all we have to do is to verify that this identification is covariant under a change of frames. To see that, let $\Xi' \in \sec(P_{\mathrm{Spin}^e_{1,3}}(M))$ be another spin frame on M. From the principal bundle structure of $P_{\mathrm{Spin}^e_{1,3}}(M)$, we know that, for each $x \in M$, there exists (a unique) $u(x) \in \mathrm{Spin}^e_{1,3}$ such that

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$\Xi'(x)=\Xi(x)u(x)$. If we define, as above, $e'_a(x)=s(\Xi'(x))\cdot\mathbf{E}_a$, then $e'_a(x)=s(\Xi(x)u(x))\cdot\mathbf{E}_a=s(\Xi(x))\mathrm{Ad}_{u(x)}\cdot\mathbf{E}_a=[(\Xi(x),\mathrm{Ad}_{u(x)}\cdot\mathbf{E}_a)]=[(\Xi(x)u(x),\mathbf{E}_a)]=[(\Xi'(x),\mathbf{E}_a)]$, which proves our claim.

Proposition 44:

$$(i)\ \mathbf{E}_a=\mathbf{1}^r_\Xi(x)e_a(x)\mathbf{1}^l_\Xi(x),\ \forall x\in M,$$

$$(ii)\ \mathbf{1}^l_\Xi\mathbf{1}^r_\Xi=1\in\mathcal{C}\ell(M,g),$$

$$(iii)\ \mathbf{1}^r_\Xi\mathbf{1}^l_\Xi=1\in\mathbb{R}_{1,3}.$$

Proof: This follows from the form of the various actions defined in Propositions 36–40. For example, for each $x\in M$, we have $\mathbf{1}^r_\Xi(x)e_a(x)=[(\Xi(x),1\mathbf{E}_a)]=[(\Xi(x),\mathbf{E}_a)]\in\sec\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ (from Proposition 38). Then, it follows from Proposition 40 that $\mathbf{1}^r_\Xi(x)e_a(x)\mathbf{1}^l_\Xi(x)=\mathbf{E}_a1=\mathbf{E}_a$, $\forall x\in M$. ■

Let us now consider how the various global sections defined above transform when the spin frame Ξ is changed. Let $\Xi'\in\sec P_{\mathrm{Spin}^e_{1,3}}(M)$ be another spin frame with $\Xi'(x)=\Xi(x)u(x)$, where $u(x)\in\mathrm{Spin}^e_{1,3}$. Let e_a, $\mathbf{1}^r_\Xi$, $\mathbf{1}^l_\Xi$ and e'_a, $\mathbf{1}^r_{\Xi'}$, $\mathbf{1}^l_{\Xi'}$ be the global sections, respectively, defined by Ξ and Ξ' (as above). We then have the following.

Proposition 45: Let Ξ,Ξ' be two spin frames related by $\Xi'=\Xi u$, where $u:M\to\mathrm{Spin}^e_{1,3}$. Then

$$\begin{aligned}&(i)\ \ e'_a=Ue_aU^{-1},\\ &(ii)\ \ \mathbf{1}^l_{\Xi'}=\mathbf{1}^l_\Xi u=U\mathbf{1}^l_\Xi,\\ &(iii)\ \ \mathbf{1}^r_{\Xi'}=u^{-1}\mathbf{1}^r_\Xi=\mathbf{1}^r_\Xi U^{-1},\end{aligned}\tag{42}$$

where $U\in\sec\mathcal{C}\ell(M,g)$ is the Clifford field associated to u by $U(x)=[(\Xi(x),u(x))]$. Also, in (ii) and (iii), u and u^{-1}, respectively, act on $\mathbf{1}^l_\Xi\in\sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ and $\mathbf{1}^r_\Xi\in\sec\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ according to Proposition 37.

Proof: (i) We have

$$\begin{aligned}e'_a(x)&=[(\Xi'(x),\mathbf{E}_a)]=[(\Xi(x)u(x),\mathbf{E}_a)]=[(\Xi(x),u(x)\mathbf{E}_au(x)^{-1})]=[(\Xi(x),u(x))]\\&\quad\times[(\Xi(x),\mathbf{E}_a)][(\Xi(x),u(x)^{-1})]=U(x)e_a(x)U(x)^{-1}.\end{aligned}\tag{43}$$

(iii) It follows from Proposition 38 that

$$\mathbf{1}^r_{\Xi'}(x)=[(\Xi'(x),1)]=[(\Xi(x)u(x),1)]=[(\Xi(x),1u(x)^{-1})]=[(\Xi(x),u(x)^{-1})]=u(x)^{-1}\mathbf{1}^r_\Xi(x),\tag{44}$$

where in the last step we used Proposition 37 and the fact that $\mathbf{1}^r_\Xi(x)=[(\Xi(x),1)]$. To demonstrate the second part, note that

$$\begin{aligned}u^{-1}(x)\mathbf{1}^r_\Xi(x)&=[(\Xi(x),u(x)^{-1})]=[(\Xi(x),1u(x)^{-1})]=[(\Xi(x),1)][(\Xi(x),u(x)^{-1})]\\&=\mathbf{1}^r_\Xi(x)U^{-1}(x),\end{aligned}\tag{45}$$

for all $x\in M$. It is important to note that in the last step we have a product between an element of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ [i.e., $[(\Xi(x),1)]$] and an element of $\mathcal{C}\ell(M,g)$ [i.e., $[(\Xi(x),u(x)^{-1})]$]. ■

We emphasize that the right unit sections associated with spin frames are *not* constant in any covariant way. In fact, we have the following.

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Proposition 46: Let $\mathbf{1}_{\Xi}^{r} \in \sec \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M)$ *be the right unit section associated to the spin frame* Ξ. *Then*

$$\nabla^{s}_{e_a} \mathbf{1}_{\Xi}^{r} = -\frac{1}{2} \mathbf{1}_{\Xi}^{r} \omega_{e_a}, \tag{46}$$

where ω_{e_a} *is the connection 1-form (Proposition 54) written in the basis* $\{e_a\}$.

Proof: It follows from Eq. (A9) of the Appendix. ∎

C. Representatives of DHSF on the Clifford bundle

Let $\{\mathbf{E}^a\}$ be, as before, a fixed orthonormal basis of $\mathbb{R}^{1,3} \subseteq \mathbb{R}_{1,3}$. Remember that these objects are fundamental to the Dirac equation (37) in terms of sections Ψ of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M,g)$:

$$D^s \Psi \mathbf{E}^{21} - qA\Psi = m\Psi \mathbf{E}^0.$$

Let $\Xi \in \sec P_{\mathrm{Spin}^{e}_{1,3}}(M)$ be a spin frame on M and define the sections $\mathbf{1}_{\Xi}^{l}$, $\mathbf{1}_{\Xi}^{r}$ and e_a, respectively, on $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M)$, $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M)$ and $\mathcal{C}\ell(M,g)$, as above. Now we can use Proposition 44 to write the above equation in terms of sections of $\mathcal{C}\ell(M,g)$:

$$(D^s \Psi) \mathbf{1}_{\Xi}^{r} e^{21} \mathbf{1}_{\Xi}^{l} - qA\Psi = m\Psi \mathbf{1}_{\Xi}^{r} e^0 \mathbf{1}_{\Xi}^{l}. \tag{47}$$

Right-multiplying by $\mathbf{1}_{\Xi}^{r}$ yields, using Proposition 44,

$$e^a (\nabla^{s}_{e_a} \Psi) \mathbf{1}_{\Xi}^{r} e^{21} - qA\Psi \mathbf{1}_{\Xi}^{r} = m\Psi \mathbf{1}_{\Xi}^{r} e^0. \tag{48}$$

It follows from Proposition 59 that

$$(\nabla^{s}_{e_a} \Psi) \mathbf{1}_{\Xi}^{r} = \nabla_{e_a}(\Psi \mathbf{1}_{\Xi}^{r}) - \Psi \nabla^{s}_{e_a}(\mathbf{1}_{\Xi}^{r}) = \nabla_{e_a}(\Psi \mathbf{1}_{\Xi}^{r}) + \frac{1}{2}\Psi \mathbf{1}_{\Xi}^{r} \omega_a, \tag{49}$$

where Proposition 46 was employed in the last step. Therefore

$$e^a \left[\nabla_{e_a}(\Psi \mathbf{1}_{\Xi}^{r}) + \frac{1}{2}\Psi \mathbf{1}_{\Xi}^{r} \omega_a \right] e^{21} - qA(\Psi \mathbf{1}_{\Xi}^{r}) = m(\Psi \mathbf{1}_{\Xi}^{r}) e^0. \tag{50}$$

Thus it is natural to define, for each spin frame Ξ, the Clifford field $\psi_{\Xi} \in \sec \mathcal{C}\ell(M,g)$ (see Proposition 39) by

$$\psi_{\Xi} := \Psi \mathbf{1}_{\Xi}^{r}. \tag{51}$$

We then have

$$e^a \left[\nabla_{e_a} \psi_{\Xi} + \frac{1}{2} \psi_{\Xi} \omega_a \right] e^{21} - qA\psi_{\Xi} = m\psi_{\Xi} e^0. \tag{52}$$

A comment about the nature of spinors is in order. As we repeatedly said in the previous sections, spinor fields should not be ultimately regarded as fields of multivectors (or multiforms), for their behavior under Lorentz transformations is not tensorial (they are able to distinguish between 2π and 4π rotations). So, how can the identification above be correct? The answer is that the definition in Eq. (51) is intrinsically spin-frame dependent. Clearly, this is the price one ought to pay if one wants to make sense of the procedure of representing spinors by differential forms.

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Note also that the covariant derivative acting on ψ_Ξ in Eq. (52) is the tensorial covariant derivative $\boldsymbol{\nabla}_V$ on $\mathcal{C}\ell(M,g)$, as it should be. However, we see from the expression above that $\boldsymbol{\nabla}_V$ always acts on ψ_Ξ together with the term $\frac{1}{2}\psi_\Xi\omega_a$. Therefore, it is natural to define an "effective covariant derivative" $\boldsymbol{\nabla}_V^{(s)}$ acting on ψ_Ξ by

$$\boldsymbol{\nabla}_{e_a}^{(s)}\psi_\Xi := \boldsymbol{\nabla}_a\psi_\Xi + \frac{1}{2}\psi_\Xi\omega_a. \tag{53}$$

Then, Proposition 54 yields

$$\boldsymbol{\nabla}_{e_a}^{(s)}\psi_\Xi = \partial_{e_a}(\psi_\Xi) + \frac{1}{2}\omega_a\psi_\Xi, \tag{54}$$

which emulates the spinorial covariant derivative, as it should. We observe moreover that if $U \in \sec \mathcal{C}\ell(M,g)$ and if $\psi_\Xi \in \sec \mathcal{C}\ell(M,g)$ is a representative of a Dirac-Hestenes spinor field then

$$\boldsymbol{\nabla}_{e_a}^{(s)}(U\psi_\Xi) = (\boldsymbol{\nabla}_{e_a}U)\psi_\Xi + U\boldsymbol{\nabla}_{e_a}^{(s)}\psi_\Xi. \tag{55}$$

(This is the derivative used in Ref. 34, there introduced in an *ad hoc* way.)

With this notation, we finally have the Dirac–Hestenes equation for the *representative* Clifford field $\psi_\Xi \in \sec \mathcal{C}\ell(M,g)$, on a Lorentzian spacetime:

$$e^a\boldsymbol{\nabla}_{e_a}^{(s)}\psi_\Xi e^{21} - qA\psi_\Xi = m\psi_\Xi e^0, \tag{56}$$

where ψ_Ξ is the representative of a DHSF Ψ of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,g)$, relative to the spin frame Ξ. (The DHE on a Riemann–Cartan spacetime will be discussed in another publication.)

Let us finally show that this formulation recovers the usual transformation properties characteristic of the Hestenes's formalism as described, e.g., in Ref. 34. For that matter, consider two spin frames $\Xi,\Xi' \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$, with $\Xi'(x) = \Xi(x)u(x)$, where $u(x) \in \mathrm{Spin}^e_{1,3}$. It follows from Proposition 45 that $\psi_{\Xi'} = \Psi\mathbf{1}^r_{\Xi'} = \Psi u^{-1}\mathbf{1}^r_\Xi = \Psi\mathbf{1}^r_\Xi U^{-1} = \psi_\Xi U^{-1}$. Therefore, the various spin frame dependent Clifford fields from Eq. (56) transform as

$$\begin{aligned} e'_a &= Ue_aU^{-1}, \\ \psi_{\Xi'} &= \psi_\Xi U^{-1}. \end{aligned} \tag{57}$$

These are exactly the transformation rules one expects from fields satisfying the Dirac–Hestenes equation (see, e.g., Ref. 34).

D. Bilinear covariants

1. Bilinear covariants associated to a DHSF

We are now in position to give a precise definition of the bilinear covariants of the Dirac theory, associated with a given DHSF.

Definition 47: Recalling that $\Lambda^p(M) \hookrightarrow \mathcal{C}\ell(M,g)$, $p=0,1,2,3,4$, *and recalling Propositions 39 and 40, the bilinear covariants associated to a DHSF* $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ *[and* $\tilde{\Psi} \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$*] are the following sections of* $\mathcal{C}\ell(M,g)$:

$$\begin{gathered} S = \Psi\tilde{\Psi} = \sigma + e_5\,\omega \in \sec(\Lambda^0(M) + \Lambda^4(M)), \\ J = \Psi\mathbf{E}_0\tilde{\Psi} \in \sec\Lambda^1(M), \quad K = \Psi\mathbf{E}_3\tilde{\Psi} \in \sec\Lambda^1(M), \end{gathered} \tag{58}$$

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$$M=\Psi \mathbf{E}_{12}\tilde{\Psi}\in \sec\Lambda^2(M),$$

where $\Psi=\Psi\frac{1}{2}(1+\mathbf{E}_0)$, *and* $e_5=e_0e_1e_2e_3$.

Remark 48: Of course, since all bilinear covariants in Eq. (58) are sections of $\mathcal{C}\ell(M,g)$, *they have the right transformation properties under arbitrary local Lorentz transformations, as required. As shown, e.g., in Ref. 21 these bilinear covariants and their Hodge duals satisfy a set of identities, called the Fierz identities (see, e.g., Ref. 34) that are crucial for the physical interpretation of the Dirac equation (in first and second quantizations).*

Remark 49: Crumeyrolle[10] *gives the name of* amorphous *spinor fields to ideal sections of the Clifford bundle* $\mathcal{C}\ell(M,g)$. *Thus an amorphous spinor field* ϕ *is a section of* $\mathcal{C}\ell(M,g)$ *such that* $\phi P=\phi$, *where* $P=P^2$ *is an idempotent section of* $\mathcal{C}\ell(M,g)$. *However, these fields and also the so-called Dirac–Kähler (Refs. 18, 20) fields, which are also sections of* $\mathcal{C}\ell(M,g)$, *cannot be used in a physical theory of fermion fields since they do not have the correct transformation law under a Lorentz rotation of the local spin frame.*

2. Bilinear covariants associated to a representative of a DHSF

We note that the bilinear covariants, when written in terms of $\psi_\Xi:=\Psi\mathbf{1}_\Xi^r$, read (from Proposition 44) as

$$S=\psi_\Xi\tilde{\psi}_\Xi=\sigma+e_5\,\omega\in\sec(\Lambda^0(M)+\Lambda^4(M)),$$

$$J=\psi_\Xi e_0\tilde{\psi}_\Xi\in\sec\Lambda^1(M),\qquad K=\psi_\Xi e_3\tilde{\psi}_\Xi\in\sec\Lambda^1(M),$$

$$M=\psi_\Xi e_1e_2\tilde{\psi}_\Xi\in\sec\Lambda^2(M),$$

where $e_5=e_0e_1e_2e_3$. These are all intrinsic quantities, as they should be.

E. Electromagnetic gauge invariance of the DHE

Proposition 50: The DHE is invariant under electromagnetic gauge transformations,

$$\psi_\Xi\mapsto\psi'_\Xi=\psi_\Xi e^{qe_{21}\chi}, \tag{59}$$

$$A\mapsto A+\partial\chi, \tag{60}$$

$$\omega_{e_a}\mapsto\omega_{e_a}, \tag{61}$$

where $\psi_\Xi,\psi'_\Xi\in\sec C\ell^0(M,g)$, $A\in\sec\Lambda^1(M)\subset\sec C\ell(M,g)$ *and where* $\chi\in\sec\Lambda^0(M)\subset\sec C\ell(M,g)$ *is a gauge function.*

Proof: It is a direct calculation. ■

But, what are the meanings of these transformations? Equation (59) looks similar to Eq. (57) defining the change of a representative of a DHSF once we change the spin frame, but here we have an active transformation, since we did *not* change the spin frame. On the other hand, Eq. (60) does not correspond either to a passive (no transformation at all) or active local Lorentz transformation for A. Nevertheless, writing $\chi=\theta/2$ yields

$$\begin{aligned}
&e^{-qe^{21}\theta/2}e^0e^{qe^{21}\theta/2}=e'^0=e^0,\\
&e^{-qe^{21}\theta/2}e^1e^{qe^{21}\theta/2}=e'^1=\cos q\theta\;\;e^1+\sin q\theta\;\;e^2,\\
&e^{-qe^{21}\theta/2}e^2e^{qe^{21}\theta/2}=e'^2=-\sin q\theta\;\;e^1+\cos q\theta\;\;e^2,\\
&e^{-qe^{21}\theta/2}e^3e^{qe^{21}\theta/2}=e'^3=e^3.
\end{aligned} \tag{62}$$

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We see that Eqs. (62) define a spin frame Ξ' to which corresponds, as we already know, a basis $\{e'^0, e'^1, e'^2, e'^3\}$ for $\Lambda^1(M) \hookrightarrow C\ell(M,g)$. We can then think of the electromagnetic gauge transformation as a rotation in the spin plane e^{21} by identifying ψ'_{Ξ} in Eq. (59) with $\psi_{\Xi'}$, the representative of the DHSF in the spin frame Ξ' and by supposing that instead of transforming the spin connection ω_{e_a} as in Eq. (A7) it is taken as fixed and instead of maintaining the electromagnetic potential A fixed it is transformed as in Eq. (60). We observe that, since in the theory of the gravitational field ω_{e_a} is associated with some aspects of that field, our interpretation for the electromagnetic gauge transformation suggests a possible nontrivial coupling between electromagnetism and gravitation, *if* the Dirac–Hestenes equation is taken as a fundamental representation of fermionic matter. We will explore this possibility in another publication.

VI. CONCLUSIONS

In this paper, we hope to have clarified the ontology of Dirac–Hestenes spinor fields [on a general spacetime $\mathfrak{M}=(M,g,\boldsymbol{\nabla},\tau_g,\uparrow)$ of the Riemann–Cartan type admitting a spin structure] and its relationship with sums of even multivector fields (or differential forms). This has been achieved through the introduction of the Clifford bundle of multivector fields $(\mathcal{C}\ell(M,g))$ and the *left* $(\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M))$ and *right* $(\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M))$ spin-Clifford bundles on a spin manifold (M,g), as well as a study of the relations among these bundles. Left algebraic spinor fields and Dirac–Hestenes spinor fields [both fields are sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$] have been defined and the relation between them has been established. Moreover, a consistent Dirac equation for a DHSF $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ (denoted DE$\mathcal{C}\ell^l$) on a Lorentzian spacetime was found. We succeeded also in obtaining in a consistent way a *representation* of the DE$\mathcal{C}\ell^l$ in the Clifford bundle. It is such equation satisfied by Clifford fields $\psi_\Xi \in \sec \mathcal{C}\ell(M,g)$ that we called the Dirac–Hestenes equation (DHE). This means that to each DHSF $\Psi \in \sec\, \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ and $\Xi \in \sec P_{\mathrm{Spin}^e_{1,3}}(M)$ there is a well-defined even nonhomogeneous multivector field $\psi_\Xi \in \sec \mathcal{C}\ell(M,g)$ (EMFS) associated with Ψ. Such a EMFS is called a *representative* of the DHSF on the given spin frame. And, of course, such a EMFS (the representative of the DHSF) is *not* a spinor field. With this crucial distinction between a DHSF and their EMFS representatives we presented a consistent theory for Clifford and spinor fields of all kinds.

We emphasize that the DE$\mathcal{C}\ell^l$ and the DHE, although related, are of different mathematical natures. This issue has been particularly scrutinized in Secs. IV and V. We studied also the local Lorentz invariance and the electromagnetic gauge invariance and showed that only for the DHE such transformations are of the same mathematical nature, something that suggests by itself a possible link between them.

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APPENDIX: COVARIANT DERIVATIVES OF CLIFFORD AND SPINOR FIELDS

1. Covariant derivative of Clifford fields

In this appendix, $(M,g,\boldsymbol{\nabla},\tau_g,\uparrow)$ denotes a general *Riemann–Cartan* spacetime (see Definition 3). Since $\mathcal{C}\ell(M,g)=\tau M/J(M,g)$, it is clear that any metric compatible $(\boldsymbol{\nabla} g=0)$ connection

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defined in τM passes to the quotient $\tau M/J(M,g)$, and thus define an algebra bundle connection.[10] In this way, the covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(M,g)$ is completely determined.

We will find formulas for the covariant derivative of Clifford fields and of DHSF using the general theory of connections in principal bundles and covariant derivatives in associate vector bundles, as described in many excellent textbooks, e.g., Refs. 8, 15, 29, 30.

Let (E,M,π_1,G,F) denoted by $E=P\times_\rho F$ be a vector bundle associated to a PFB (P,M,π,G) by the linear representation ρ of G in $F=\mathbf{V}$.

Definition 51: Let $\sigma:\mathbb{R}\supset I\to M$, $t\mapsto\sigma(t)$ be a curve in M with $x_0=\sigma(0)\in M$, and let $p_0 \in \pi^{-1}(x_0)$. The parallel transport of p_0 along σ is given by the curve $\hat{\sigma}:\mathbb{R}\supset I\to P, t\mapsto\hat{\sigma}(t)$ defined by

$$\frac{d}{dt}\hat{\sigma}(t)=\Gamma_p\left(\frac{d}{dt}\sigma(t)\right), \tag{A1}$$

with $p_0=\hat{\sigma}(0)$ and $\pi(\hat{\sigma}(t))=\sigma(t)$. We also denote $p_{\|t}=\hat{\sigma}(t)$.

In Eq. (A1), $\Gamma_p:T_xM\to T_pP$ is a connection on (P,M,π,G) (see, e.g., definition (a) on p. 358 of Ref. 8).

Consider the trivializations of P,

$$\Phi_i:\pi^{-1}(U_i)\to U_i\times G, \quad \Phi_i(p)=(\pi(p),\phi_{i,x}(p)),$$

and E,

$$\Xi_i:\pi_1^{-1}(U_i)\to U_i\times F, \quad \Xi_i(q)=(\pi_1(q),\chi_i(q))=(x,\chi_i(q)).$$

Then, we have the following.

Definition 52: The parallel transport *of $\Psi_0\in E$, $\pi_1(\Psi_0)=x_0$, along the curve $\sigma:\mathbb{R}\supset I\to M$, $t\mapsto\sigma(t)$ from $x_0=\sigma(0)\in M$ to $x=\sigma(t)$ is the element $\Psi_{\|t}\in E$ such that the following occurs*:

(i) $\pi_1(\Psi_{\|t})=x$,
(ii) $\chi_i(\Psi_{\|t})=\rho(\phi_i(p_{\|t})\circ\phi_i(p_0)^{-1})\chi_i(\Psi_0)$.
(iii) $p_{\|t}\in P$ is the parallel transport of $p_0\in P$ along σ from x_0 to x.

Definition 53: Let v be a vector at x_0 tangent to the curve σ (as defined above). The covariant derivative of $\Psi\in\sec E$ along v is denoted $(D_v^E\Psi)_{x_0}\in\sec E$ and

$$(D_v^E\Psi)(x_0)\equiv(D_v^E\Psi)_{x_0}=\lim_{t\to 0}\frac{1}{t}(\Psi_{\|t}^0-\Psi_0), \tag{A2}$$

where $\Psi_{\|t}^0$ is the parallel transport of the vector $\Psi_t\equiv\Psi(\sigma(t))$ of the given section $\Psi\in\sec E$ along σ from $\sigma(t)$ to x_0. The only requirements on σ are that $\sigma(0)=x_0$ and

$$\left.\frac{d}{dt}\sigma(t)\right|_{t=0}=v. \tag{A3}$$

Proposition 54: Let $V\in\sec TM$. The covariant derivative of a Clifford field $A \in\sec\mathcal{C}\ell(M,g)$ is given by

$$\boldsymbol{\nabla}_V A=V(A)+\frac{1}{2}[\omega_V,A], \tag{A4}$$

where $V(A):=V(A^I)e_I$ and ω_V is the connection 1-form $V\mapsto\omega_V=-\frac{1}{2}V^a\Gamma_{abc}e^b\wedge e^c$, written in the basis $\{e_a\}$, with Γ_{abc} given by $\boldsymbol{\nabla}_{e_a}e_b=\Gamma_{ab}{}^c e_c=\Gamma_{abc}e^c$.

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Proof: Writing $A(t)=A(\sigma(t))$ in terms of the multivector basis $\{e_I\}$ of sections associated to a given spin frame, as in Sec. V B, we have $A(t)=A^I(t)e_I(t)=A^I(t)[(\Xi(t),E_I)]=[(\Xi(t),A^I(t)E_I)]=[(\Xi(t),a(t))]$, with $a(t):=A^I(t)E_I\in\mathbb{R}_{1,3}$. If follows from item (ii) of Definition 52 that

$$A^0_{\|t}=[(\Xi(0),g(t)a(t)g(t)^{-1})], \tag{A5}$$

for some $g(t)\in\mathrm{Spin}^e_{1,3}$, with $g(0)=1$. Thus

$$\lim_{t\to 0}\frac{1}{t}(g(t)a(t)g(t)^{-1}-a(0))=\left[\frac{dg}{dt}ag^{-1}+g\frac{da}{dt}g^{-1}+ga\frac{dg^{-1}}{dt}\right]_{t=0}$$
$$=\dot a(0)+\dot g(0)a(0)-a(0)\dot g(0)=V(A^I)E_I+[\dot g(0),a(0)],$$

where $\dot g(0)\in\mathrm{Lie}(\mathrm{Spin}^e_{1,3})=\Lambda^2(\mathbb{R}^{1,3})$. Therefore

$$\boldsymbol{\nabla}_V A=V(A^I)e_I+\frac{1}{2}[\omega_V,A],$$

for some $\omega_V\in\sec\Lambda^2(M)$. In particular, calculating the covariant derivative of the basis 1-vector fields e_b yields $V^a\Gamma_{ab}{}^c e_c=\boldsymbol{\nabla}_V e_b=\frac{1}{2}[\omega_V,e_b]$, so that $\omega_V=-\frac{1}{2}V^a\Gamma_{abc}e^b\wedge e^c$. ■

Remark 55: Equation (A4) shows that the covariant derivative preserves the degree of a homogeneous Clifford field, as can be easily verified.

The general formula Eq. (A4) and the associative law in the Clifford algebra immediately yields the following.

Corollary 56: The covariant derivative $\boldsymbol{\nabla}_V$ on $\mathcal{C}\ell(M,g)$ acts as a derivation on the algebra of sections, i.e., for $A,B\in\sec\mathcal{C}\ell(M,g)$ and $V\in\sec TM$, it holds that

$$\boldsymbol{\nabla}_V(AB)=(\boldsymbol{\nabla}_V A)B+A(\boldsymbol{\nabla}_V B). \tag{A6}$$

Under a change of gauge (local Lorentz transformation) $e^a\mapsto e'^a=Ue^aU^{-1}$, with $U\in\sec\mathcal{C}\ell(M,g), U\tilde U=\tilde U U=1$, the corresponding transformation law for ω_V is as follows.

Corollary 57: Under a change of gauge (local Lorentz transformation), ω_V transforms as

$$\frac{1}{2}\omega_V\mapsto U\frac{1}{2}\omega_V U^{-1}+(\boldsymbol{\nabla}_V U)U^{-1}. \tag{A7}$$

Proof: It is a simple calculation using Eq. (A4). ■

2. Covariant derivatives of spinor fields

The spinor bundles introduced in Sec. II, like $I(M)=P_{\mathrm{Spin}^e_{1,3}}(M)\times_\ell I$, $I=\mathbb{R}_{1,3}\frac{1}{2}(1+E_0)$, and $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$, $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$ (and subbundles) are examples of vector bundles. Thus, the general theory of covariant derivative operators on associate vector bundles can be used (as in the previous section) to obtain formulas for the covariant derivatives of sections of these bundles. Given $\Psi\in\sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ and $\Phi\in\sec\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$, we denote the corresponding covariant derivatives by $\boldsymbol{\nabla}^s_V\Psi$ and $\boldsymbol{\nabla}^s_V\Phi$. [Recall that $I^l(M)\hookrightarrow C\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ and $I^r(M)\hookrightarrow C\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$.]

Proposition 58: Given $\Psi\in\sec\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ and $\Phi\in\sec\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$,

$$\boldsymbol{\nabla}^s_V\Psi=V(\Psi)+\frac{1}{2}\omega_V\Psi, \tag{A8}$$

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$$\boldsymbol{\nabla}_V^s \Phi = V(\Psi) - \frac{1}{2}\Psi\,\omega_V. \tag{A9}$$

Proof: It is analogous to that of Proposition 54, with the difference that Eq. (A5) should be substituted by $\Psi^0_{\|t} = [(\Xi(0), g(t)a(t))]$ and $\Phi^0_{\|t} = [(\Xi(0), a(t)g(t)^{-1})]$. ■

Proposition 59: Let $\boldsymbol{\nabla}$ *be the connection on* $\mathcal{C}\ell(M,g)$ *to which* $\boldsymbol{\nabla}^s$ *is related. Then, for any* $V \in \sec TM$, $A \in \sec \mathcal{C}\ell(M,g)$, $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$) *and* $\Phi \in \sec \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)$,

$$\boldsymbol{\nabla}_V^s (A\Psi) = A\boldsymbol{\nabla}_V^s \Psi + (\boldsymbol{\nabla}_V A)\Psi, \tag{A10}$$

$$\boldsymbol{\nabla}_V^s (\Phi A) = \Phi\boldsymbol{\nabla}_V A + (\boldsymbol{\nabla}_V^s \Phi) A. \tag{A11}$$

Proof: Recalling that $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ $[\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M)]$ is a module over $\mathcal{C}\ell(M,g)$, the result follows from a simple computation. ■

Finally, let $\Psi \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M)$ be such that $\Psi\mathbf{e} = \Psi$ where $\mathbf{e}^2 = \mathbf{e} \in \mathbb{R}_{1,3}$ is a primitive idempotent. Then, since $\Psi\mathbf{e} = \Psi$,

$$\boldsymbol{\nabla}_V^s \Psi = \boldsymbol{\nabla}_V^s (\Psi\mathbf{e}) = V(\Psi\mathbf{e}) + \frac{1}{2}\omega_V \Psi\mathbf{e} = \left[V(\Psi) + \frac{1}{2}\omega_V \Psi \right]\mathbf{e} = (\boldsymbol{\nabla}_V^s \Psi)\mathbf{e}, \tag{A12}$$

from where we verify that the covariant derivative of a LIASF is indeed a LIASF.

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